May 2020 Discrete Mathematics Qualifying Exam

Closed book, closed notes, no calculators.

1) (20 points) Prove using mathematical induction that

$$\sum_{i=0}^{n-1}(3i+2) = (3n+1)n/2 \text{ for all integers } n \geq 1.$$

Note: You have to use mathematical induction; if you prove the statement using any other method, it will not count.

Solution:

Proof (by mathematical induction):

Base case:

The statement is true for n=1 because LHS = $0+2=2$ and RHS = $\frac{(3+1)}{2} = 2$.

Inductive step:

Suppose that for some integer $k \geq 1$, $\sum_{i=0}^{k-1}(3i+2) = \frac{(3k+1)k}{2}$. This is the inductive hypothesis.

We must show that $\sum_{i=0}^{k+1-1}(3i+2) = \frac{(3(k+1)+1)(k+1)}{2}$.

LHS = $\sum_{i=0}^{k+1-1}(3i+2)$

= $\left(\sum_{i=0}^{k-1}(3i+2)\right) + (3(k+1-1)+2)$ by separating the last term

= $(3k+1)k/2 + (3k+2)$ by inductive hypothesis

= $\frac{3k^2+k+6k+4}{2} = \frac{3k^2+7k+4}{2}$ by algebra

RHS = $\frac{(3(k+1)+1)(k+1)}{2} = \frac{(3k+4)(k+1)}{2} = \frac{3k^2+7k+4}{2}$.

LHS = RHS. Therefore, the statement is true for all integers $n \geq 1$.

2) (a) (8 points) Rewrite the following statement in if-then form:

"Leaving the house before 7:30am is a necessary condition for my being on time for work"

(b) (8 points) Is the following conditional statement true or false? Explain your answer.

"If the Sun rotates around the Earth then 2 plus 2 is 5."

Solution:

a) "If I am on time for work then I must have left the house before 7:30am"
b) "the Sun rotates around the Earth" is false and "2 plus 2 is 5" is false. Conditional statement false →false is true.

3) (16 points) Evaluate (compute the value of) the following summations:

(a) (6 points) $\sum_{k=1}^{6}(k+1)$

(b) (10 points) $\sum_{i=3}^{n-1} 2^i$, where n≥4.

Solution:

(a) $\sum_{k=1}^{6}(k+1)$ = 2+3+4+5+6+7 = 27

(b) $\sum_{i=3}^{n-1} 2^i = 2^3 + 2^4 + \cdots + 2^{n-1} = 2^3(1 + 2 + 2^2 + \cdots + 2^{n-4}) = 2^3 \frac{2^{n-3}-1}{2-1}$

$$= 2^3(2^{n-3} - 1) = 2^n - 8$$

4)

a) (8 points) Is {{a,d,e},{b,c},{d,f}} a partition of {a,b,c,d,e}? Explain.

b) (8 points) Suppose X={1,2} and Y={a,b,c}. Find the Cartesian product of X and Y (list all elements of the Cartesian product X × Y).

Solution:

a) No. Because d appears in two sets: {a,d,e} and {d,f}.
b) X × Y = {(1,a),(1,b),(1,c),(2,a),(2,b),(2,c)}

5) (a) (6 points) How many functions (total functions) f are there from the set {a, b, c} to the set {1, 2, 3, . . . n} such that f(a) = f(b)?

Note: In some textbooks they use terms "total function" and "partial function". If you studied using such textbook, then we mean "total functions" in this question.

In other textbooks they only use term "functions". If you studied with the textbook where only term "function" was used, then we mean "functions" in this question.

(b) (5 points) How many one-to-one functions (total functions) are there from a set of 3 elements to a set of 8 elements?

(c) (5 points) How many onto functions (total functions) are there from a set of 3 elements to a set of 8 elements?

Solution:

(a) f(a) can be any of 1, 2, …, n (n choices). f(b) must be equal to f(a) (1 choice). f(c) can be any of 1, 2, …, n (n choices). Total, there are n^2 functions.

(b) The first element can be mapped to any of the 8 elements. The second element must be mapped to a different element than the first one was mapped (7 choices). The third element must be mapped to a different element than the first and the second are mapped (6 choices). Total there are $8 \cdot 7 \cdot 6$ one-to-one functions.

(c) Three elements can have at most 3 different images so not every element of the 8-element set will have an element that maps into it. Therefore, the number of onto functions is 0.

6) (16 points) Prove by contradiction that there is no least positive rational number. Recall that a number is rational if and only if it can be expressed as a quotient of two integers with a nonzero denominator.

Solution.

Assume the contrary. Assume, there is a least positive rational number p. By definition of rational, $p=a/b$, where a and b are integers and b is not 0. Let us consider the number $q=a/(2b)$. Number q is rational because a and 2b are integers. Since p is positive, q is also positive and $q<p$ because $q = p/2$. In other words, q is a positive rational number than is less than p. This contradicts the assumption that p is the least positive rational number. Therefore, there is no least positive rational number.